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Torsion cycles as non-local magnetic sources in non-orientable spaces

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ABSTRACT

Non-orientable spaces can appear to carry net magnetic charge, even in the absence of magnetic sources. It is shown that this effect can be understood as a physical manifestation of the existence of torsion cycles of codimension one in the homology of space.

1. Introduction

The idea that electromagnetic charge may be a manifestation of the topology of space has a very long history [Whe, WM, Sor]. The basic configuration considered by Wheeler and Misner was a space that contained a handle-like region, often called a wormhole. This space is traversed by sourceless electric lines of force that enter the wormhole radially through one mouth and exit through the other one some distance away. For an observer situated sufficiently far from the wormhole, space appears approximately flat, but not sourceless: the two mouths look like two pointlike electric charges.

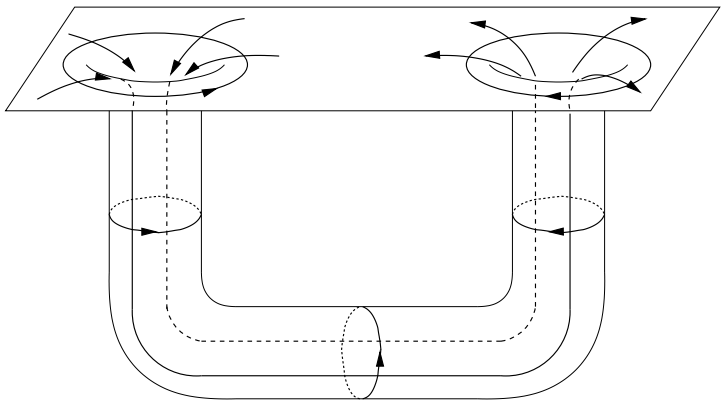

Fig.1: An orientable handle

A "dual" construction can be imagined in which the lines of force are oriented circles that surround the wormhole mouths and extend to its interior in a continuous way, as sketched in Fig. 1. From a distance, the two mouths appear to carry dual charges of opposite signs. As a matter of terminology, we will refer to the dual lines of force, and the charges they

correspond to, as magnetic, although the motivation for doing that will only become clear later on.

Serious objections can be raised against Misner and Wheeler's wormhole as a realistic model of electromagnetic charge, on the grounds that wormholes are not satisfactory solutions in General Relativity. Under fairly general assumptions, the null energy condition has to be violated at some points near the wormhole throat [HV]. This indicates that wormhole solutions require the coupling of gravity to some type of exotic matter. Even if the right type of matter and coupling is found, known wormhole solutions are very unstable. Charged wormholes also exist in General Relativity in the presence of scalar fields non-minimally coupled to gravity. The scalar field provides a good example of the type of exotic matter required, but these charged wormholes are still unstable under spherically symmetric perturbations [BG].

Leaving aside these objections to the Misner-Wheeler idea of charge, the strength of either of the two apparent charges would be defined by the flux through a sphere $\mathcal{S}$ that surrounds one of the mouths of the wormhole. The size of the sphere is of course immaterial, because the underlying field is divergenceless, and all such spheres are crossed by the same amount of flux and hence define the same charge strength. On the other hand, $\mathcal{S}$ has to be large enough to be recognisably outside the wormhole throat. Unfortunately, the construction given in [Whe, MW] lacks an intrinsic distinction between the part of space that belongs to the wormhole, and that part that remains "outside" of it. There is no clear way to tell when the sphere $\mathcal{S}$ is far away enough for its flux to be unambiguously attributable to one of the mouths of the wormhole. These ambiguities are due to the wormhole in [Whe, MW] being regarded as a purely topological feature of space, lacking any geometric structure.

Another insatisfactory feature of the Misner-Wheeler construction is that it is unable to accommodate the idea of an individual isolated charge. Any such charge has to be understood as half of a dipole, the other half of which is removed far away from it. Besides, the throat that joins the two apparent charges establishes a correlation between them that seems difficult to reconcile with quantum mechanics [Sor]. For these reasons, the Misner-Wheeler wormhole can never be taken seriously as a realistic model of charged particles.

Some time later, Sorkin constructed a non-orientable version of the Wheeler and Misner wormhole, and showed that the effect of the non-orientability was that the flux that emanates from one of the ends of the handle becomes reversed [Sor]. To a distant observer, Sorkin's non-orientable handle appears to be a source of net flux. This construction provides a model of individual, isolated charges that is based purely upon the topology of space. These ideas were further elaborated and extended in [DH], who generalised Sorkin's proposal and provided new illustrations of it.

In [Sor, DH], the non-orientability of space was assumed to be concentrated in a small region of space, and to be bounded by a sphere S. Outside of S, space remains orientable. These authors showed that there exist field configurations where the flux of the magnetic pseudovector field $\mathbf{B}$ through S was nonzero, even though $\mathbf{B}$ has zero divergence. This was interpreted as the presence a new kind of magnetic charge situated inside $\mathcal{S}$.

Fig. 2 gives a two-dimensional illustration of this idea. A non-orientable surface is crossed

by magnetic lines of force, and the non-orientability causes the lines of force to emerge with the same orientation out of both ends. A distant observer would assign the same magnetic charge to both of them. In contrast, electric flux lines would have behaved very much as in Fig. 1, and each end would have seemed to carry opposite electric charges. This example suggests that the non-orientability affects only one kind of charge, namely the one we called dual or magnetic.

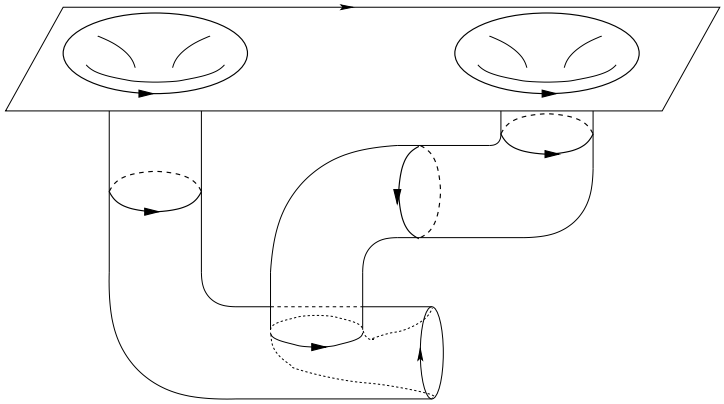

Fig.2: A non-orientable handle

It is not clear if the non-orientable spaces introduced by Sorkin can be obtained as solutions to General Relativity or, if they could, whether they would be stable. As with the original idea of Misner and Wheeler, there is no scale with which to make sense of the statements that the non-orientable region of space is small, and that the observer who measures the flux through $\mathcal{S}$ must be far away from that region. In the absence of any known answers to these questions, we will ignore them, and focus instead in the purely topological aspects of these constructions.

The ideas that will be discussed in the main body of this paper are similar to those in [Sor], although the point of view will be rather different. None of the objections pointed out earlier will be answered, and it will not be claimed that these models can provide an alternative model of electromagnetic charge. Rather, the goal will be to show that a different connection between magnetic charge and non-orientability can be found, in which the topological properties of non-orientable spaces appear more explicitly.

The idea of magnetic charge that we will consider is defined by the flux of the electromagnetic field strength F through a sphere $\mathcal{S}$ that bounds a non-orientable region of space. Being differential forms, not densities, neither F nor dF can be integrated on non-orientable spaces. The flux of F through $\mathcal{S}$ is not problematic, since spheres are always orientable, but the integral of dF over the non-orientable interior of $\mathcal{S}$ is not defined. This precludes the use of the Stokes theorem for differential forms, and for this reason the connection between the flux of F and any notion of magnetic charge located inside $\mathcal{S}$ cannot be of a conventional type. We will propose a way to make sense of the Stokes theorem in these circumstances, and show that a notion of magnetic charge associated with the flux of F can still be made to work.

Once that is done, we will investigate the topological properties of non-orientable manifolds

that make possible the existence of that kind of magnetic charge. It is well known that non-orientable manifolds are characterised by the existence of a torsion cycle of codimension one. More exactly, if $\mathcal{N}$ is the non-orientable interior of $\mathcal{S}$, then the homology group $H_{m-1}(\mathcal{N}, \mathcal{S})$ contains one and only one torsion subgroup that is cyclic of order two. Using that result, we will show that the homology class of the torsion cycle determines the value of the magnetic charge carried by $\mathcal{N}$, as measured by the flux of F through $\mathcal{S}$.

2. First notions

We shall be working in an $(m+1)$-dimensional spacetime of the form $\mathbf{R} \times \mathcal{K}$, where the real line $\mathbf{R}$ is the time direction, and the smooth m-manifold $\mathcal{K}$ represents space. By construction, this spacetime will be time-orientable, so it will be orientable if and only if the space manifold $\mathcal{K}$ is. In this spacetime there is a closed $(m-1)$-form F, the electromagnetic field strength. The presence of magnetic or electric sources in spacetime is expressed by the dual currents μ_m and μ_e, which are related to the field strength F and its Hodge dual $*F$ by the equations,

$$\begin{aligned} dF &= \mu_m, \\ d*F &= \mu_e. \end{aligned} \tag{2.1}$$

The two m-form μ_m and μ_e are taken to be purely spatial, *i.e.* they will have no timelike components. This is a physical assumption based on the fact that the world-volumes of localised charged sources are always timelike. The dual current, being transversal to the world-volume, must then be always spacelike.

Assuming that the Stokes theorem for differential forms can be applied, (2.1) relate the magnetic and electric charges to the fluxes of F and $*F$ respectively. Suppose that $\mathcal{N}$ is an m-dimensional region of $\mathcal{K}$ whose frontier is a smoothly embedded $(m-1)$-sphere $\mathcal{S}$. Then the magnetic charge contained in $\mathcal{N}$,

$$Q_m(\mathcal{N}) = \int_{\mathcal{N}} \mu_m, \tag{2.2}$$

equals the flux of F through $\mathcal{S}$,

$$Q_m(\mathcal{N}) = \int_{\mathcal{N}} \mu_m = \int_{\mathcal{N}} dF = \int_{\mathcal{S}} F.$$

Similarly, if ν is a 3-dimensional region of $\mathcal{K}$ bounded by a sphere σ, the electric charge contained in ν, defined as

$$Q_e(\nu) = \int_{\nu} \mu_e, \tag{2.3}$$

equals the flux of $*F$ through σ,

$$Q_e(\nu) = \int_{\nu} \mu_e = \int_{\nu} d*F = \int_{\sigma} *F.$$

Although charges can be defined directly in terms of the currents, as in (2.2) and (2.3), we will only be concerned with those charges that can be expressed as fluxes. The general question of whether charges are always fluxes or not depends on a rather intrincate interplay between the topology of the world-volume of the sources and the topology of the spacetime manifold in which they propagate. For a thorough analysis of this question in compact orientable manifolds with boundary, see [AO].

3. Global considerations

The equations (2.1), which hold locally at every point in space, leave open the possibility that F or $*F$, or even the dual currents μ_m and μ_e may not be globally defined. Suppose that we cover the whole space $\mathcal{K}$ with sets $\{U_i\}$ such that both the individual sets U_i and their multiple intersections are topologically spheres. An electrically charged scalar particle is represented by a complex wave function $\psi(x)$, which can be viewed as a vector in an internal two-dimensional space. Given a choice of orthonormal basis in that internal space, its components $\psi_1(x)$ and $\psi_2(x)$ correspond to a definition of real and imaginary parts of $\psi(x)$. The internal basis can be chosen in many different ways at every point x, and any two choices are related by an element of the group $O(2)$.

In general, each set U_i is equipped with its own continuous choice of internal bases. If x is a point in the overlap $U_i \cap U_j$, the two conventions are related by a transition function $g_{ij}(x) \in O(2)$. This situation is indicated by saying that the internal symmetry group $O(2)$ has been gauged.

The reflection element of $O(2)/SO(2)$ reverses the sign of $\psi_2(x)$ and is thus the charge conjugation operator. If a transition function g_{ij} equals the reflection element, then $\psi(x)$ in U_i becomes its complex conjugate $\psi^*(x)$ in U_j. Physically, a particle whose electric charge is positive in the conventions of U_i, is regarded as having negative charge in the conventions of U_j. We then say that there is no global notion of positive charge. This problem is eliminated in $U_i \cap U_j$ by reversing the orientation of the internal frames in, say, U_j, in which case the transition function becomes an element of the subgroup $SO(2) \subset O(2)$. This, however, may only shift the problem over to other intersections. A global notion of positive charge can always be restored in simply-connected spaces, but may be impossible in a space that is not simply-connected. When that happens, the particle could start a travel around some non-contractible cycle of space as a positive charge, and return as a negative charge. This is analogous to transporting a frame once along a non-contractible loop in a Möbius band.

In a situation like this, no global definition of electric charge or current is possible. Nevertheless, it can be shown that observable quantities, *e.g.* the electromagnetic forces that acts on an electrically charged particle, are independent of these conventions, so that all observers would agree in their measurements. Another surprising property of these spaces is that they can hold net electric charge even when they are compact, *i.e.*, lacking a boundary through which the lines of force could escape. Although it is impossible to attribute a definite sign to it globally, the charge will be non-zero to any observer. These results were first described by Kiskis [Kis], who also extended these ideas to non-abelian gauge theories.

We shall assume that the field strength $(m-1)$-form F is global. Hence, the above mentioned effects are excluded. Nevertheless, the assumption that the space manifold is non-orientable has similar consequences for the magnetic charge as the twist in the internal $O(2)$ space had for the electric charge. In our model, the dual magnetic current μ_m is an m-form on the m-manifold $\mathcal{K}$. Consequently, it must be proportional to $\Omega_{\mathcal{K}}$, the volume form of $\mathcal{K}$,

$$\mu_m = C(w)\Omega_{\mathcal{K}}, \tag{3.1}$$

in which $C(w)$ is a distribution localised on w, the world-line of the magnetic charge. The equation (3.1) only makes sense globally if $\mathcal{K}$ is orientable. If it is not, $\Omega_{\mathcal{K}}$ can only be defined locally in the sets U_i that cover the space manifold $\mathcal{K}$. There will then exist a loop λ along which the $\Omega_{\mathcal{K}}$ chosen at one point gets transported back into $-\Omega_{\mathcal{K}}$, and that would be interpreted as the magnetic charge having changed sign after traversing λ. This is a purely classical effect that resembles the reversal of electric charge that occurs when the $O(2)$ bundle that acts on the wave function cannot be reduced to an $SO(2)$ bundle [Kis].

Because we are assuming that F is global, the same must be true of μ_m. In what follows, $\mathcal{K}$ will be non-orientable, and so, as discussed, μ_m can only be defined globally up to a sign. This leaves $\mu_m = 0$ as the only possible choice, which makes F a closed form, *i.e.*, the first equation in (2.1) reduces to $dF = 0$. It follows from this that no nonzero single-valued magnetic charge, as defined in (2.2), can exist in $\mathcal{K}$.

Another implication of $\mathcal{K}$ being non-orientable is that, because there is no global volume element with which to define the Hodge star, the dual field strength $*F$ cannot exist globally. It will reverse sign whenever $\Omega_{\mathcal{K}}$ does, as was the case when it was transported around the loop λ. Therefore μ_e will also have to change sign upon traversing λ. This purely classical effect caused by the non-orientability of space resembles the quantum reversal of charge described in [Kis].

In conclusion, whithin the framework of the equations of motion (2.1), spacetimes of the form $\mathbf{R}\times\mathcal{K}$ where the spatial manifold $\mathcal{K}$ is non-orientable cannot support non-zero single-valued electric or magnetic currents, and hence neither electric nor magnetic charges in the sense of (2.2) and (2.3).

It will be shown below that, despite the exclusion of magnetic sources implied by the condition $dF = 0$, it is still possible to define a different kind of magnetic charge, understood as a flux of F through a sphere in $\mathcal{K}$, which owes its existence precisely to the non-orientability of the space manifold $\mathcal{K}$.

4. Fluxes, orientations, and charges

Given the closed differential $(m-1)$-form F and a smoothly embedded $(m-1)$-sphere S in $\mathcal{K}$, the magnetic flux $\Phi(\mathcal{S})$ is defined as

$$\Phi(\mathcal{S}) = \int_{\mathcal{S}} F. \tag{4.1}$$

Although $\mathcal{S}$ has been chosen to be a sphere, the definition (4.1) still makes sense if $\mathcal{S}$ is allowed to be any orientable $(m-1)$-cycle in $\mathcal{K}$. The assumption of orientability is needed in order to integrate the differential form F on $\mathcal{S}$. In connection to this, it should be pointed out that there exists an alternative definition of magnetic flux which involves only densities, and is thus applicable to any $(m-1)$-cycle in $\mathcal{K}$, orientable or not. The question arises of whether that alternative definition is equivalent to (4.1). It will turn out that the answer depends on whether the interior of $\mathcal{S}$ is orientable or not.

In order to explain this other notion of magnetic flux and its relationship with (4.1), we temporarily assume that our spacetime $\mathbf{R} \times \mathcal{K}$ is orientable, with volume form Ω and associated Hodge dual $*$. Then the magnetic field is defined intrinsically as the vector field $\mathbf{B}$ such that

$$F = dx^0 \wedge E + \mathrm{i}_{\mathbf{B}} \mathrm{i}_0 \Omega. \tag{4.2}$$

The $(m-2)$-form E is the generalisation of the electric field to $(m+1)$ spacetime dimensions when F is an $(m-1)$-form, as we are assuming. The contraction of the timelike unit vector field and the volume form, $\mathrm{i}_0\Omega$, coincides whith the spatial volume form $\Omega_{\mathcal{K}}$ introduced in (3.1),

$$\Omega_{\mathcal{K}} = \mathrm{i}_0 \Omega.$$

Substituting the decomposition (4.2) into $dF = 0$ shows that the divergence of $\mathbf{B}$ is zero,

$$\operatorname{div} \mathbf{B} = 0. \tag{4.3}$$

This result has been obtained under the assumption that the space manifold $\mathcal{K}$ is orientable. Nevertheles, we will keep (4.3) as part of the definition of a sourceless magnetic field even is space is non-orientable.

The relationship between $\mathbf{B}$ and F is roughly represented by the magnetic lines of force and their duals shown in Figs. 1 and 2. In those two-dimensional examples, $\mathbf{B}$ is a vector field and F a one-form which are represented by the magnetic lines of force and their duals respectively. In our framework, $\mathbf{B}$ remains a vector field whatever the dimension of space, and for that reason the magnetic lines of force are always lines in any dimension. In contrast, the rank of F is one less than the dimension of space, so that the dual lines of force are in general spacelike hypersurfaces of codimension one.

It may be worth pointing out that the intrinsic definition of $\mathbf{B}$ given in (4.2) involves a choice of volume form Ω. In particular, switching $\Omega_{\mathcal{K}}$ to $-\Omega_{\mathcal{K}}$ amounts to changing $\mathbf{B}$ into $-\mathbf{B}$. This circumstance is familiar from vector calculus, where it appears as the choice of the "right-hand rule" in the definition of the cross product and the curl. The dependence of $\mathbf{B}$ upon the orientation of space is expressed by saying that $\mathbf{B}$ is a *pseudovector* field.

Remembering that $\mathcal{S}$ is spacelike, (4.1) becomes

$$\Phi(\mathcal{S}) = \int_{\mathcal{S}} \mathrm{i}_{\mathbf{B}} \Omega_{\mathcal{K}}.$$

Because $\mathcal{S}$ is orientable and $\mathcal{K}$ has been assumed, for the time being, to be orientable as well, it follows that $\mathcal{S}$ is also coorientable. Then the volume form $\Omega_{\mathcal{K}}$ induces a volume form $\Omega_{\mathcal{S}}$ in $\mathcal{S}$,

$$\Omega_{\mathcal{S}} = \mathrm{i}_{\mathbf{n}}\Omega_{\mathcal{K}}$$

in which $\mathbf{n}$ is a unit vector normal to $\mathcal{S}$. Then the flux $\Phi(\mathcal{S})$ can be rewritten in terms of the pseudovector field $\mathbf{B}$ and the unit vector $\mathbf{n}$ as follows,

$$\begin{aligned}\Phi(\mathcal{S}) &= \int_{\mathcal{S}} \mathrm{i}_{\mathbf{B}}\Omega_{\mathcal{K}} = \int_{\mathcal{S}} \mathrm{i}_{(\mathbf{B}\cdot\mathbf{n})\mathbf{n}}\Omega_{\mathcal{K}} \\ &= \int_{\mathcal{S}} (\mathbf{B}\cdot\mathbf{n})\mathrm{i}_{\mathbf{n}}\Omega_{\mathcal{K}} = \int_{\mathcal{S}} (\mathbf{B}\cdot\mathbf{n})\Omega_{\mathcal{S}}.\end{aligned}$$

The volume element $\Omega_{\mathcal{S}}$ carries an associated measure in $\mathcal{S}$ that we denote by ds, and so

$$\Phi(\mathcal{S}) = \int_{\mathcal{S}} (\mathbf{B}\cdot\mathbf{n})\, ds. \tag{4.4}$$

Independently of its origin in (4.1), the expression (4.4) makes sense for any $(m-1)$-cycle $\mathcal{S}$ in $\mathcal{K}$ whether it is orientable or not, so long as it possesses the required normal unit vector $\mathbf{n}$, that is, so long as $\mathcal{S}$ is coorientable in $\mathcal{K}$. This will always be true whenever both $\mathcal{S}$ and $\mathcal{K}$ are orientable, as we have been assuming since (4.2). Under those assumptions, (4.1) and (4.4) are, as we have shown, equivalent.

We will be mostly interested in the case where $\mathcal{K}$ is non-orientable. Then, $\Omega_{\mathcal{K}}$ can only be defined locally and, through (4.2), the same will be true of $\mathbf{B}$. In this situation, the relationship between (4.1) and (4.4) has to be reexamined. It will be useful to keep in mind that orientability is an intrinsic property of a space, whereas coorientability is not, because it depends on the way it is embedded in a larger space. For example, a loop can be embedded in a Klein bottle in two topologically inequivalent ways, only one of which makes the loop coorientable, whereas the loop itselt is always orientable. In general, an orientable submanifold $\mathcal{S}$ embedded in $\mathcal{K}$ is coorientable only if it has an orientable tubular neighbourhood [BT].

Suppose then that $\mathcal{S}$ is orientable and that it has an orientable tubular neighbourhood in $\mathcal{K}$. Then $\Phi(\mathcal{S})$ can be defined as either the flux of F or as the flux of $\mathbf{B}$, making use of a volume form defined in the tubular neighbourhood of $\mathcal{S}$. Nevertheless, because the volume form around $\mathcal{S}$ cannot be extended to all of $\mathcal{K}$, the $\mathbf{B}$ constructed out of it cannot be extended either. It is not a proper divergenceless pseudovector field. In that sense, only (4.1) provides a satisfactory definition of magnetic charge. Alternatively, we could have said that $\mathbf{B}$ is the fundamental object that exists globally, and that the spacelike part of F is constructed out of it once a volume form is chosen. From that point of view, F may fail to exist globally in a non-orientable space owing to the lack of a globally defined volume form. In that case, the correct definition of magnetic charge would have been (4.4). In

short, (4.1) and (4.4) cease to be equivalent whenever F and $\mathbf{B}$ cannot be globally defined simultaneously.

In what follows, we will adopt the view that F is the fundamental object, and that it is globally defined unless stated otherwise. Then, (4.1) will be our primary notion of magnetic flux, whereas (4.4) will be a secondary result, lacking significance if $\mathbf{B}$ fails to exist globally.

4-1 Magnetic charge

Let us now add the extra assumption that $\mathcal{S}$ separates the space manifold $\mathcal{K}$ into two parts that we call $\mathcal{M}$ and $\mathcal{N}$. The part $\mathcal{M}$ will be non-compact and orientable, whereas $\mathcal{N}$ is compact but may or may not be orientable and has $\mathcal{S}$ as its only frontier. An $\mathcal{N}$ non-orientable will be interpreted below as a quasi-localised region of space whose topology is capable of displaying magnetic charge, as first suggested by Sorkin [Sor]. Configurations of this type can be constructed by taking the connected sum of two m-manifolds, one non-compact and orientable and the other closed and non-orientable. The connected sum consists in removing the interior of an embedded m-ball from each manifold, and pasting the remainders together by means of an homeomorphism on the boundary spheres of these balls. The two boundary spheres become the single sphere $\mathcal{S}$.

By construction, $\mathcal{S}$ will always have an orientable neighbourhood, and hence will be both orientable and coorientable. Therefore, both definitions of magnetic flux, (4.1) and (4.4) are in principle applicable to it. In order to emphasise the conceptual difference between the two, we will introduce a specific notation for the latter:

$$\begin{aligned} \Phi(\mathcal{S}) &= \int_{\mathcal{S}} F, \\ \mathcal{B}(\mathcal{S}) &= \int_{\mathcal{S}} (\mathbf{B}\cdot\mathbf{n})\, ds. \end{aligned} \tag{4.5}$$

If there is net magnetic flux of the first or second kind through $\mathcal{S}$, we will say that $\mathcal{N}$ carries magnetic F-charge or B-charge respectively. As has already been explained, these definitions are only satisfactory when the globally well-defined object is F or $\mathbf{B}$ respectively.

In order to be able to obtain general results concerning the values of $\Phi(\mathcal{S})$ and $B(\mathcal{S})$, we need to make use of the Stokes theorem, which relates those fluxes to the behaviours of F or $\mathbf{B}$ in the interior of $\mathcal{N}$. In fact, there are two versions of the Stokes theorem, depending on whether we are working with differential forms, or with densities [AMR]. Integrating differential forms requires an orientation with which to turn the volume element into a measure. Hence, the Stokes theorem for differential forms is only applicable to orientable manifolds, and in particular we need $\mathcal{N}$ to be orientable. If so, then

$$\int_{\mathcal{N}} dF = \int_{\mathcal{S}} F. \tag{4.6}$$

Remembering that F is exact, it follows that an orientable region $\mathcal{N}$ never carries magnetic F-charge,

$$\mathcal{N} \text{ orientable} \quad \Longrightarrow \quad \Phi(\mathcal{S}) = 0. \tag{4.7}$$

On the other hand, the Stokes theorem for densities is applicable whether $\mathcal{N}$ is orientable or not, because densities can be integrated directly without any extra requirements. If we depart for a moment from our view that F must always be globally defined, and assume that $\mathbf{B}$ extends smoothly to the interior of $\mathcal{N}$ (at the expense of F if $\mathcal{N}$ is non-orientable), then

$$\int_{\mathcal{N}} \operatorname{div} \mathbf{B}\, dv = \int_{\mathcal{S}} (\mathbf{B} \cdot \mathbf{n})\, ds, \tag{4.8}$$

where dv is the measure in the interior of $\mathcal{N}$. But the condition (4.3) implies that the region $\mathcal{N}$, orientable or not, never carries magnetic B-charge either,

$$B(\mathcal{S}) = 0. \tag{4.9}$$

In summary, the two notions of magnetic charge defined in (4.5) agree when $\mathcal{N}$ is orientable, and both vanish. When $\mathcal{N}$ is non-orientable and $\mathbf{B}$ exists globally, then the B-charge is well-defined but also vanishes, whereas the F-charge is not well-defined. The only remaining possibility is that $\mathcal{N}$ is still non-orientable, and F globally defined. The question is then whether the F-charge can ever be non-zero, and what its values can be.

The Stokes theorem for differential forms (4.6) is inapplicable when $\mathcal{N}$ is non-orientable, so that no conclusions can be drawn from it. Nevertheless, we will find a way to make sense of (4.6) for $\mathcal{N}$ non-orientable by means of a cell decomposition, as will be explained below. This procedure will relate the values of $\Phi(\mathcal{S})$ to certain topological quantities intrinsic to $\mathcal{N}$, independent of the choice of cell decomposition, and hence endowed with an invariant meaning.

4-2 Remarks

Before explaining how (4.6) can be made to work for $\mathcal{N}$ non-orientable, let us first discuss what could in principle be wrong with it. The right-hand side still makes sense, because $\mathcal{S}$ is orientable, and so F can be integrated on it. In contrast, the left-hand side contains the m-form dF integrated on the non-orientable m-manifold $\mathcal{N}$, an operation that does not make sense. It is this that makes the Stokes theorem for differential forms unusable in non-orientable manifolds.

That problem notwithstanding, we must recall that we are not dealing with a generic differential form, but with a closed one. There is no difficulty in integrating $dF = 0$ on any m-manifold, orientable or not, and so the left-hand side of (4.6), being always zero, still makes sense after all.

Even if each side of (4.6) makes sense individually when $dF = 0$ whether $\mathcal{N}$ is orientable or not, the equation itself is only valid for $\mathcal{N}$ orientable. Nevertheless, we can think of $\mathcal{N}$ as

the union of a number of m-dimensional cells, and (4.6) is applicable in each of the cells. In the next section, we will show that this reinterpretation of the manifold as a cell complex leads to a version of (4.6) that is valid for non-orientable manifolds with frontier. Its left-hand side will still be required to be zero, but its right-hand side will receive a correction that turns the frontier of $\mathcal{N}$ into the more general concept of homology boundary.

The difference between frontier and homology boundary is fundamental in what follows. It will be useful to give a short review of these ideas, before analysing their relevance to the value of the magnetic F-charge.

5. Two notions of boundary

In point-set topology, an m-manifold with boundary is locally mapped to $R_+ \times R^{m-1}$, and the boundary is the submanifold that gets mapped into $0 \times R^{m-1}$. That conforms to the ordinary idea of boundary as a frontier where the manifold ends, and in fact we have been referring to this notion as "frontier" in previous paragraphs.

From the point of view of homology, an m-manifold can be regarded as a sum of m-cells, each cell being homeomorphic to an m-ball. The m-cells can be visualised as m-dimensional polyhedra that fill out the entire manifold without overlapping. The boundary of an m-cell, defined as in point-set topology, *i.e.*, as its frontier, is now regarded as the sum of its faces. These faces now are the $(m-1)$-cells in our cell decomposition of the manifold, and so on.

The collection of all k-cells is known as the k-skeleton of the manifold, where k runs from 0 to m. We will adopt the notation V_a^k for the elements of the k-skeleton, where a is an appropriate index that labels its elements. Because the cells V_a^k are homeomorphic to k-balls, the boundaries ∂V_a^k are homeomorphic to $(k-1)$-spheres.

The precise way in which the boundaries of the $(r+1)$-cells fit into the r-skeleton is summarised in the incidence matrix $M_{ab}^{r+1,r}$, defined by

$$\partial V_a^{r+1} = \sum_b M_{ab}^{r+1,r} V_b^r . \tag{5.1}$$

The entries of these matrices can only be 0 or ± 1. As a matter of terminology, we will say that V_a^{r+1} incides upon V_b^r whenever $M_{ab}^{r+1,r} \neq 0$.

The operation of taking the boundary of a cell is required to be nilpotent, that is, it must always be the case that $\partial^2 = 0$. At the level of cells, this requirement follows from the earlier statement that ∂V_a^m is homeomorphic to the sphere S^{m-1}, and hence boundaryless. In order to reflect this nilpotency, the incidence matrices are required to satisfy the relationship

$$\sum_b M_{ab}^{r+1,r} M_{bc}^{r,r-1} = 0. \tag{5.2}$$

In the definition of the m-dimensional cells V_a^m, there is implied a choice of orientation for each one of them. Recall that the m-cells are homeomorphic to m-balls, and for these the

notion of orientation is straightforward. Then, just as for m-balls, there are two choices possible for the orientation of each cell. In principle there is no preferred way to choose all the orientations. Nevertheless, some choices are better than others, in a sense to be explained below. For the time being, we will assume that every m-cell has been given a definite orientation. This induces an orientation on its boundary in the usual way. Once this is done, the integration of differential forms is defined both in the interior of the cells and on their boundaries.

Now, we identify the manifold with the sum of all the m-cells,

$$\mathcal{N} = \sum_a V_a^m. \tag{5·3}$$

When we combine this identification between the manifold and its cell decomposition with the definition (5.1) of the incidence matrix, the boundary of $\mathcal{N}$ can be given as

$$\partial\mathcal{N} = \sum_a \partial V_a^m = \sum_a \sum_b M_{ab}^{m,m-1} V_b^{m-1}. \tag{5·4}$$

Every $(m-1)$-cell in the cell decomposition of a manifold is the face of one or two m-cells. In contrast, cells of dimension $m-2$ and lower can be shared by any number of higher-dimensional cells, depending on the details of the cell decomposition. It follows that the colums of the incidence matrix $M_{ab}^{m,m-1}$ can be of three types:

$$\overset{V^{m-1}_{a,\text{ type I}}}{\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}} \quad \text{or} \quad \overset{V^{m-1}_{a,\text{ type II}}}{\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}} \quad \text{or} \quad \overset{V^{m-1}_{a,\text{ type III}}}{\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}, \tag{5·5}$$

up to overall signs that can be arranged to be as shown. The first type corresponds to sets $V^{m-1}_{a,\text{ type I}}$ that lie entirely on the frontier of $\mathcal{N}$. The second and third types corresponds to cells $V^{m-1}_{a,\text{ type II}}$ or $V^{m-1}_{a,\text{ type III}}$ that are shared by two different m-dimensional cells. The difference between these two types becomes apparent when we use (5.4) to calculate $\partial\mathcal{N}$. All the type II cells cancel out, and we are left with

$$\partial\mathcal{N} = \sum_a V^{m-1}_{a,\text{ type I}} + 2\sum_a V^{m-1}_{a,\text{ type III}}. \tag{5·6}$$

This formula illustrates the difference between the notions of boundary used in homology theory and point-set topology. In our notation, the latter comprises all the type I cells, and nothing else; as a subspace of $\mathcal{N}$, it coincides with the frontier $\mathcal{S}$. The former includes also a certain $(m-1)$ cycle formed from all of the type III cells, multiplied by two. It will be explained below that the appearence of this cycle is due to the existence of a torsion subgroup in the homology group $H_{m-1}(\mathcal{N},\mathcal{S};\mathbf{Z})$ that is cyclic of order two, whenever $\mathcal{N}$ is non-orientable.

Lacking a standard terminology, we will use the name "free boundary" for the point-set boundary, and the name "torsion boundary" for the contribution of the torsion $(m-1)$-cycles. The "homology boundary" is the sum of both. The symbol $\partial\mathcal{N}$ will be reserved for the homology boundary of $\mathcal{N}$, and so we write

$$\partial\mathcal{N} = \sum_a V^{m-1}_{a,\text{ type I}} + 2\sum_a V^{m-1}_{a,\text{ type III}} = (\partial\mathcal{N})_{\text{free}} + (\partial\mathcal{N})_{\text{torsion}}. \tag{5.7}$$

As a chain, the free boundary is the sum of all type I cells, and so it manifestly depends upon the choice of cell decomposition. As a subspace of $\mathcal{N}$, however, it does not, because, as has already been pointed out, it coincides with the frontier $\mathcal{S}$. In contrast, the torsion boundary does depend on that choice, even as a subspace of $\mathcal{N}$. It will now be shown that, if $\mathcal{N}$ is orientable, then there always exists a cell decomposition for which there is no torsion boundary.

6. Orientability and cell decompositions

When we calculate $\partial\mathcal{N}$ using an arbitrary cell decomposition, the result is generally of the form

$$\partial\mathcal{N} = (\partial\mathcal{N})_{\text{free}} + 2\gamma, \tag{6.1}$$

where γ is made entirely of the type III cells of dimension $m-1$. The torsion boundary is the term 2γ, and γ is the torsion cycle. This last formula shows that the double cycle 2γ is homologous to minus the free boundary of $\mathcal{N}$. In the language of relative homology, γ is a relative torsion $(m-1)$-cycle.

It is known that the torsion subgroup of $H_{m-1}(\mathcal{N},\mathcal{S};\mathbf{Z})$ is zero if $\mathcal{N}$ is orientable (see *e.g.* [Mas]). In that case, γ itself must be a relative boundary, say $\gamma = \partial\alpha - \beta$ for some chains $\alpha \subset \mathcal{N}$ and $\beta \subset \mathcal{S}$. Then,

$$\partial(\mathcal{N} - 2\alpha) = (\partial\mathcal{N})_{\text{free}} - 2\beta.$$

This shows that our cell decomposition had the wrong signs in the regions α and β. Reversing those signs provides a corrected cell decomposition that is more natural in the sense that the $\partial\mathcal{N}$ calculated with its help does not introduce any spureous torsion boundaries. These are the "better" choices we mentioned earlier in relation to (5.3).

The situation for non-orientable manifolds is different. No choice of cell decomposition can eliminate the torsion boundary, although it can change it by a homology. In these

circumstances, it is useful to know that the frontier of any manifold, orientable or not, is always orientable. Then we can begin by orientating the frontier, that is, the type I $(m-1)$-cells, and then provide the adjoining m-cells with matching orientations. There is no natural prescription for orientating the rest of the cells.

These ideas are best illustrated by means of an example. The Möbius band is a non-orientable 2-manifold with boundary. It can be obtained from the 2-dimensional projective plane by cutting out an open disk, which leaves behind a one-dimensional frontier. A cell decomposition of the Möbius band that only uses two 2-cells is shown in Fig. 3. An orientation was first chosen for the frontier, and then the cells C_1 and C_2 were orientated acordingly.

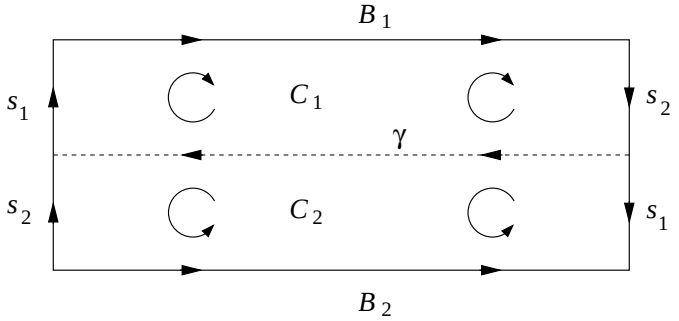

Fig.3: A cell decomposition of the Möbius band

In the notation shown in the figure, the boundaries of the two cells are

$$\begin{aligned} \partial C_1 &= s_1 + s_2 + B_1 + \gamma, \\ \partial C_2 &= -s_1 - s_2 + B_2 + \gamma. \end{aligned} \tag{6.2}$$

This can be rewritten in terms of an incidence matrix as follows:

$$\partial \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 0 & 1 \\ -1 & -1 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ B_1 \\ B_2 \\ \gamma \end{pmatrix}.$$

From the incidence matrix we see that B_1 and B_2 (columns 3 and 4) are type I cells, s_1 and s_2 (columns 1 and 2) are type II, and γ (column 5) is type III. Then, B_1 and B_2 form a cell decomposition of the frontier of the Möbius band and γ is a torsion cycle. The boundary of the Möbius band is given by

$$\partial \mathcal{N} = \partial(C_1 + C_2) = B_1 + B_2 + 2\gamma, \tag{6.3}$$

which is of the general form (6.1), with $B_1 + B_2$ the free boundary $\mathcal{S}$, and 2γ the torsion boundary.

7. Stokes theorem for cell decompositions

We are now in a position to apply the Stokes theorem for differential forms (4.6) to a closed $(m-1)$-form F that exists in a compact m-manifold $\mathcal{N}$, not necessarily orientable, with non-empty frontier $\mathcal{S}$. We start by introducing a cell decomposition of the manifold, as in (5.3). Because each m-cell has been provided with an orientation, (4.6) can be applied to it,

$$\int\limits_{V_a^m} dF = \int\limits_{\partial V_a^m} F. \tag{7.1}$$

Recalling that F is closed, this shows that

$$\int\limits_{\partial V_a^m} F = 0. \tag{7.2}$$

Let us consider what will happen when the contribution of all the m-cells is added. Depending on whether the orientations of two adjoining m-cells agree or not, the $(m-1)$-cells that separate them will be type II or III. The orientations induced on a type II cell by those of the two m-cells that incide on it will be opposite. Therefore, F is integrated twice on every type II cell, but with opposite orientations. The two integrals will cancel out, and so we find that type II cells make no contribution to the sum. In contrast, the induced orientations will agree on all type III cells, so that the two fluxes will add up instead of cancelling out.

Finally, the contribution of type I cells is straightforward: it is the flux of F on the free boundary of $\mathcal{N}$, that is, on its frontier $\mathcal{S}$. In fact, the way the fluxes of F combine when all the contributions from the cells are added reproduces the calculation of the homology boundary of $\mathcal{N}$, as defined in (5.4),

$$\sum_a \int\limits_{\partial V_a^m} F = \int\limits_{\partial \mathcal{N}} F.$$

When this is combined with (7.2), the result is what we regard as the Stokes theorem for closed differential forms, in a format that is applicable whether $\mathcal{N}$ is orientable or not,

$$\int\limits_{\partial \mathcal{N}} F = 0. \tag{7.3}$$

It has already been explained that, if $\mathcal{N}$ is orientable, then $\partial\mathcal{N}$ coincides with its frontier $\mathcal{S}$, in which case (7.3) is a repetition of (4.7). On the other hand, if $\mathcal{N}$ is non-orientable, $\partial\mathcal{N}$ consists of two distinct parts, namely the free and torsion boundaries, as in (6.1), and so we find that

$$\int\limits_{\mathcal{S}} F = -2 \int\limits_{\gamma} F. \tag{7.4}$$

The left-hand side of this is $\Phi(\mathcal{S})$, the magnetic F-charge contained inside the sphere $\mathcal{S}$, defined in (4.5). The result (7.4) shows that the F-charge need not be zero, as we now discuss.

8. Discussion

It was found in (4.9) that the magnetic B-charge contained in $\mathcal{N}$ will always vanish, whether $\mathcal{N}$ is orientable or not, so long as $\mathbf{B}$ exists globally inside $\mathcal{N}$ and remains divergenceless. On the other hand, the result (7.4) shows that, if $\mathcal{N}$ is non orientable, it may contain a non-zero magnetic F-charge.

$$\Phi(\mathcal{S}) = -2 \int_{\gamma} F. \tag{8.1}$$

This shows that the presence of the torsion cycle γ allows the magnetic flux of F through $\mathcal{S}$ to be nonzero. Because the torsion subgrop of $H_{m-1}(\mathcal{N}, \mathcal{S}; \mathbf{Z})$ is cyclic of order two, γ will always exists, and be unique up to homology. In fact, (8.1) depends only on the homology class of γ, owing to the closedness of F,

$$\int_{\gamma+\partial C} F = \int_{\gamma} F + \int_{C} dF = \int_{\gamma} F.$$

This indicates that the cycle γ, or rather its homology class $[\gamma]$, is a peculiar kind of magnetic source. In contrast with a magnetic current, which is localised at the world-line of a magnetic charge, the homology class $[\gamma]$ is not a localised object, and for that reason it is not meaningful to ask for a spacetime picture of how exactly the flux appears in $\mathcal{N}$ and then crosses the sphere $\mathcal{S}$. A rough representation of the process can be given only if a choice of γ is made, and this is illustrated in Fig. 4 for a Möbius strip. In that figure, the frontier of the Möbius strip was renamed $\mathcal{S}$ in accordance with our notation for bounding spheres.

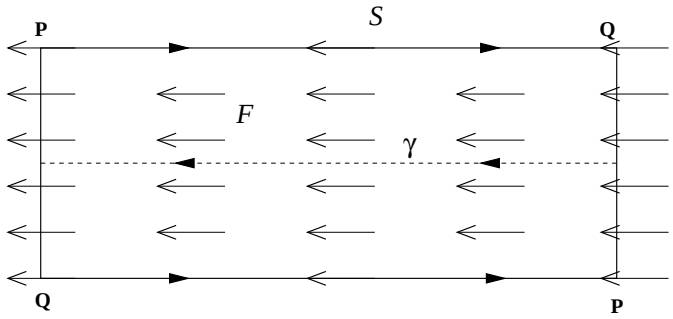

Fig.4: A torsion cycle acting as a magnetic source

A non-zero value for the right-hand side of (8.1) would be interpreted as $\mathcal{N}$ holding a sort of magnetic charge that does not emanate from a localised source, but rather from an intrinsic topological property of $\mathcal{N}$. Because F is a closed $(m-1)$-form, it represents some cohomology class in $H^{m-1}(\mathcal{N}; \mathcal{S})$. Then, a non-zero value for $\Phi(\mathcal{S})$ requires the cohomology group $H^{m-1}(\mathcal{N}; \mathcal{S})$ to be also non-zero.

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